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THE FIRM IN GENERAL EQUILIBRIUM

THEORY

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1. Introduction

In classical theory, from Smith to Mill, fixed coefficients in production are assumed. In such a context, the individual firm plays little role in the general equilibrium of the economy. The scale of any one firm is indeterminate, but the demand conditions determine the scale of the industry and the demand by the industry for inputs. The firm's role is purely passive, and no meaningful boundaries between firms are established. No doubt the firm or the entrepeneur was much discussed and indeed given a central role in the informal parts of the discussion; the role was that of overcoming disequilibria. When profit rates were unequal, profit-hungry entrepreneurs moved quickly, with the end result of eliminating their functions.

When Walras first gave explicit formulation to the grand vision of general equilibrium, he took over intact the fixed-coefficient assumptions and therewith the passive nature of the firm. In the last quarter of the nineteenth century, J. B. Clark, Wicksteed, Barone, and Walras himself recognized the possibility of alternative production activities in the form of the production function. However, so long as constant returns to scale were assumed, the size of the firm remained indeterminate. The firm did have now, even in equilibrium, a somewhat more active role than in earlier theory; it at least had the responsibility of minimizing costs at given output levels.

There were other economists, however, who were interested in the theory of the firm as such, the earliest being Cournot (1838). Anyone with an elementary knowledge of calculus and a theory that firms are maximizing profits under competitive conditions is led without thinking to the hypothesis of increasing marginal costs or diminishing returns to scale. As Cournot also knew, firms may be monopolists as well as competitors; and in those circumstances, profit maximization is compatible with increasing returns to scale.

As in other aspects of economics, both of these somewhat contradictory tendencies appear in Marshall's welter of imprecise insights. It would be tedious to follow the subsequent discussions of laws of return and their relation to competitive or other equilibrium, carried on intermittently by such authors as Wicksell, Pareto, Robertson, Shove, and Viner (with the famous assistance of Y. K. Wong). Among the literary economists in the Anglo-American tradition, a kind of orthodoxy has emerged, in the U-shaped cost curve for the firm plus free entry. In more modern language, the production possibility set of the typical firm displays an initial tendency toward increasing returns followed at higher scales by decreasing returns. The first phase is explained by indivisibilities, the second by the decreasing ability of the entrepreneur to control the firm. As one may put it, entrepreneurship should also be regarded as an input to the firm; then, after the initial phase at least, the firm would have constant returns to all inputs (including entrepreneurship) but, since by definition the firm has only one entrepreneur, there are diminishing returns to all other factors. (The indivisibility of the entrepreneur is sometimes invoked to explain the initial phase also, though of course there are typically also indivisibilities of a more definitely technological variety.) The assumption of free entry implies that the supply of entrepreneurship in the economy is infinite, or, more precisely that it is sufficiently large that its demand price will fall to zero at a point at which supply still exceeds demand.

The exact relation of this model of the firm to a full general equilibrium model has never been explored; in particular, the notion of an infinite supply of entrepreneurship is no more reasonable than that of an infinite supply of anything else.

The first mathematical model of general equilibrium was the work of Wald, summarized in Wald [1936], though some of the basic considerations in the model were suggested to him by K. Schlesinger [1933-7]. In Wald's work, to the extent that production was involved at all, fixed coefficients were assumed. After the mathematical tools available had been greatly improved by von Neumann and others as part of the development of game theory, more general models were developed by McKenzie [1954] and Arrow and Debreu [1954]. The best systematic account is that of Debreu [1959]; detailed improvements are found in Debreu [1962] and a somewhat different viewpoint in McKenzie [1959].

The treatment of the firm in Arrow-Debreu is unchanged in Debreu's later work. The set of firms is regarded as fixed. It should be noted, though, that a firm might find it most profitable to produce nothing; hence, what is ordinarily called entry here appears as a change from zero to positive output levels. The production possibility sets of the firms are assumed to be convex. This assumption excludes the possibility of an initial phase of increasing returns; it is compatible with either constant or diminishing returns to scale. The treatment of entrepreneurship in the model can then be interpreted in several ways. The most natural is to assume that entrepreneurship *per se* is not included in the list of commodities. Then where there are constant returns, entrepreneurship is not a factor of production, or, alternatively, it is not scarce. However, diminishing returns plus a finite fixed set of (potential) firms imply scarcity of entrepreneurship and positive pure profits. In this interpretation, too, we are not constrained to identify entrepreneurship as being supplied by any particular set of individuals; the diminishing returns can inhere in the operating properties of the organization.

Alternatively, we can assume that entrepreneurial resources are included among the list of commodities and are supplied by specific individuals. This is McKenzie's assumption [1959]; he completes it naturally by assuming constant returns to scale in all commodities. Then firms are distinguished by their needs for specific entrepreneurial resources (it is not assumed that entrepreneurship for one firm is necessarily the same as for another) and are limited in scale by the limitations on these resources.

The two models differ in their implications for income distribution. The Arrow-Debreu model creates a category of pure profits which are distributed to the owners of the firm; it is not assumed that the owners are necessarily the entrepreneurs or managers. Since profit maximization is assumed, conflict of interest between the organization or its management, on the one hand, and the owners on the other is assumed always to be resolved in favor of the owners. The model is sufficiently flexible, however, to permit the managers to be included among the owners.

In the McKenzie model, on the other hand, the firm makes no pure profits (since it operates at constant returns); the equivalent of profits appears in the form of payments for the use of entrepreneurial resources, but there is no residual category of owners who receive profits without rendering either capital or entrepreneurial services.

Several writers, especially Farrell [1959] and Rothenberg [1960], have argued that "small" non-convexities, such as a limited initial phase of increasing returns, are compatible with an "approximate" equilibrium, i.e., one in which discrepancies between supply and demand are small relative to the size of the market. Hence, the U-shaped cost curve is not basically incompatible with competitive general equilibrium theory, though so far there has been no rigorous development of the relations.

Substantial increasing returns to the firm, on the other hand, are obviously incompatible with the existence of a perfectly competitive equilibrium. It is of course in situations like this that monopolies arise. The theory of the profit-maximizing monopoly in a single market was developed in its essentials by Cournot and has been developed further only on secondary points, the most important of which has been the possibility of price discrimination. But, apart from some remarks in Pareto, the first serious discussions of monopoly in a general-equilibrium context are those of J. Robinson [1933] and Chamberlain [1933]. The formulation of an explicit model of general equilibrium with monopolistic elements will be discussed in section 3 below.

In static theories of general equilibrium and in the absence of monopoly, then, the individual firm has been characterized by diminishing returns, a phenomenon associated with the vague concept of entrepreneurship. Kalecki [1939, Chapter 4] suggested long ago that the reasons for limitation on the size of the firm might be found in dynamic rather than static considerations. Recent years have been the beginning of dynamic analysis of the firm (especially Penrose [1959] and Marris [1964]). From the point of view of realism and of interpretation of observations, these are a major advance. But on the production side they still retain the basic structure of the static model, restated in dynamic terms. Specifically, while returns to scale are constant in the long run , there are diminishing returns to the rate of growth, which plays the same role as scale does in a static model. (This view of attaching costs to rates of change has also been urged by some of those close to operations research; see, i.a., Hoffman and Jacobs [1954], Holt, Modigliani, Muth and Simon [1960, pp. 52-53]; Arrow, Karlin, and Scarf [1958, p. 22].) Hence, the analysis of stationary states of the dynamic system has strong formal resemblance to purely static analysis; or, to put the matter the other way, static analysis remains useful provided it is interpreted parabolically rather than literally.

However, dynamic analysis may have deeper implications if we depart from the analysis of stationary states. The firm must now serve some additional roles. In the absence of futures markets, the firm must serve as a forecaster and as a bearer of uncertainty. Further, from a general equilibrium point of view, the forecasts of others become relevant to the evaluation of the firm's shares and therefore possibly of the firm's behavior. The general equilibrium to be analyzed is, in the first instance, the equilibrium of a moment, <u>temporary equilibrium</u> in the terminology of Hicks.

Some of these topics will be discussed below; for others, only open questions can be mentioned. The analysis will always concern itself with the existence of equilibrium under each of varying sets of assumptions. Existence of equilibrium is of interest in itself; certainly a minimal property a model purporting to describe an economic system ought to have its consistency. In practice, the development of conditions needed to insure the existence of equilibrium turn out in many cases to be very revealing; until one has to construct an existence proof, the relevance of many of these conditions is not obvious.

The proofs will not be presented in detail, but their general outlines will be indicated. In section 2, a sketch will first be given of a proof of existence of competitive equilibrium under standard assumptions. In section 3, a model of monopolistic competitive equilibrium will be presented and analyzed for existence; this will display the role of the firm as price-maker. In section 4, the existence of temporary equilibrium and its precondtions are discussed. More detailed proofs of the r sults of these sections will be found in Arrow and Hahn [forthcoming, Chapter VI, sections 4 and 3, respectively].

2. The Existence of General Competitive Equilibrium.

Since proofs of existence of equilibrium in more extended contexts start from the methods used in the perfectly competitive case, it is indispensable to indicate the main lines of the proof in that case. Although it would doubtless be possible to use the proofs of Debreu or McKenzie (cited above) as starting points, I have in fact used a new form of the proof which will appear in Arrow and Hahn [forthcoming, Chapters III- V].

First, we list the assumptions made. Production is assumed organized in firms; let Y_f be the production possibility set for firm f, with typical element $\underset{\sim}{y}_f$.

I. Y_f is a closed convex set, and $\underset{\sim}{0}$ belongs to Y_f.

The last clause means that a firm can go out of existence.

II. If $\sum_f \underset{\sim}{y}_f \geq 0$ and y_f belongs to Y_f, all f, then $\underset{\sim}{y}_f = 0$, all f.

(To assert that a vector is non-negative means that each element is non-negative.) To see the meaning of II, note first that if $\sum_f \underset{\sim}{y}_f \geq 0$ but not $\sum_f \underset{\sim}{y}_f = 0$, then the productive sector as a whole is supplying positive amounts of some goods with no inputs, a physical impossibility. If $\sum_f y_f = 0$ but not all y_f's are 0, then some firms are in effect undoing the productive activity of others. If we assume that there are some inputs such as labor that are not produced by any firm, then such cancellation is impossible.

In view of II, production is possible only if the economy has some initial supply of non-produced commodities; let $\bar{x}$ be this vector of initial endowments. We now assume that with the initial endowment it is possible to have a positive net output of all commodities, that is, we can use part but less than all of each initially available commodity to produce something of each produced commodity after netting out interindustry flows.

III. It is possible to choose $\bar{y}_f$ from Y_f for each f so that the net output vector, $\sum_f \bar{y}_f + \bar{x}$, has positive components for all commodities.

Among the three production assumptions, really only the convexity assumed in I can be regarded as dubious.

By a _production allocation_ will be meant a specification of $y_f \in Y_f$ for each f. By a _feasible production allocation_ will be meant a production allocation which does not require more net inputs than are available from the initial endowment:

$$\sum_f y_f + \bar{x} \geq 0.$$

Then it is possible to demonstrate from I and II that,

the set of feasible production allocations is convex, closed and bounded.

To discuss the assumptions about consumers, let X_h be the set of consumption vectors possible to household h. For present purposes, it can simply be regarded as the set of all non-negative vectors where leisure is taken as one good. (A somewhat more complicated description is required to take care of the possibility that an individual may be capable of offering more than one kind of labor.)

IV. X_h is closed and convex and contains only non-negative vectors. Each household is assumed to possess some part of society's initial endowment, say $\underset{\sim}{\bar{x}}_h$. A somewhat technical assumption is needed to insure that in a certain sense households can make choices without any trade and even without using all of whatever initial endowment they possess.

V. For each h, there exists $\bar{\bar{x}}_h \in X_h$, such that $0 \leq \underset{\sim}{\bar{\bar{x}}}_h \leq \underset{\sim}{\bar{x}}_h$; further, any positive component of $\underset{\sim}{\bar{x}}_h$ is also a positive component of $\bar{x}_h - \bar{\bar{x}}_h$.

If we take X_h to be the set of non-negative vectors, then $\underset{\sim}{\bar{\bar{x}}}_h$ can be taken equal to $\underset{\sim}{0}$.

The final assumption about the consumer is the usual one about the continuity and convexity of consumer preferences.

VI. The preferences of household h can be represented by a continuous utility function, $U_h(x_h)$, with the following convexity property (referred to as <u>semi-strict quasi concavity</u>): if x_h^1 and x_h^2 are consumption vectors such that $U_h(x_h^1) > U_h(x_h^2)$ and if α is a scalar, $0 < \alpha \leq 1$, then $U_h[\alpha x_h^1 + (1-\alpha) x_h^2] > U_h(x_h^2)$. Further, assume that there is no satiation in all commodities simultaneously, i.e., for every $x_h^1 \varepsilon X_h$, there exists $x_h^2 \varepsilon X_h$ for which $U_h(x_h^2) > U_h(x_h^1)$.

The convexity condition implies that indifference surfaces are convex but not necessarily strictly so (thus, they may possess flat segments); however, there are no "thick" bands in which all sufficiently close vectors are indifferent. Permitting flat segments on the indifference surfaces is necessary if one is to avoid assuming that all commodities enter directly into each household's utility function. The non-satiation condition is consistent with satiation in any specific commodity or group of commodities.

Assumption VI is restrictive, but the consequences of dropping it do not appear to be severe. If it is assumed that the endowment of no household is large relative to total endowment, then the discontinuities of individual household demand functions relative to the economy as a whole are small, and so the Farrell-Rothenberg argument shows that equilibrium is approximately attained; a fully rigorous version for the case of a pure exchange economy is to be found in Starr [1969].

Finally, we need an assumption about the relation between the initial endowment held by a household and the possibility of its improving someone's welfare. A given household, h', holds some commodities initially in positive amount and others in in zero amount. Call that set of commodities the <u>h'-assets</u>. Now consider any allocation of resources to firms and households which is feasible for the given endowment vector. Such an allocation defines a <u>utility allocation</u>, a specification of the utility level of each household. Now suppose that some increase in society's endowment of h'-assets, all other components of the initial endowment remaining constant, permits a new resource allocation in which every household is at least as well off and household h" better off. If this improvement is possible starting from any feasible allocation, then household h' is said to be <u>resource-related</u> to household h".

A weaker relation between two households is the following: household h' is said to be <u>indirectly resource-related</u> to household h" if there exists some chain of households, beginning with h' and ending with h", such that each household in the chain is resource-related to its successor. We now assume,

VII. Every household is indirectly resource-related to every other. (This definition is related to, but not identical with, that of irreducibility of the economy in McKenzie [1959] and [1961], and it generalizes assumptions introduced by Arrow and Debreu [1954, pp. 279-281, assumptions VI and VII].)

This assumption is very weak; each household is assumed to have something to offer the market which is valuable to someone, who in turn is similarly linked to someone else, and so forth till everyone is reached. Certainly in an advanced economy, it can easily be accepted.

The income of the household, available for its consumption, derives in general from two sources: the sale of its endowment and its share of the profits of firms. Since we are assuming convexity but not necessarily constant returns to scale, it is possible for firms to have positive profits even at equilibrium. It is therefore assumed that each household h has the right to a share, d_{hf}, in the profits of firm f. Necessarily,

$$d_{hf} \geq 0, \ \sum_h d_{hf} = 1 \text{ for all f.} \tag{1}$$

Then the income of the household is defined by,

$$M_h = \underset{\sim}{p}\, \bar{\underset{\sim}{x}}_h + \sum_f d_{hf}\, (\underset{\sim}{p}\, \underset{\sim}{y}_f), \tag{2}$$

since the profits of firm f are defined by $\underset{\sim}{p}\, \underset{\sim}{y}_f$.

We now state formally the usual definition of competitive equilibrium.

D. 1. A price vector, p* and an allocation (x^*_h, y^*_f) constitute a competitive equilibrium if the following conditions are satisfied:

(a) $\underset{\sim}{p}^* \geq 0$ but $\underset{\sim}{p}^* \neq 0$;

(b) $\sum_h x_h^* \leq \sum_h \bar{x}_h + \sum_f y_f^*$;

(c) y_f^* maximizes $p^* y_f$ subject to $y_f \varepsilon Y_f$;

(d) x_h^* maximizes $U_h(x_h)$ subject to $x_h \varepsilon X_h$, $p^* x_h$

$$\leq p^* \bar{x}_h + \sum_f d_{hf} (p^* y_f^*) = M_h^*.$$

It turns out that the demand functions of the consumer defined implicitly by (d) can be discontinuous if prices approach a limit at which $M_h^* = 0$. It is convenient to first introduce a slightly different and weaker definition of competitive equilibrium, prove its existence, and then show that under the assumptions made (particularly VII) it also satisfies the conditions of D.1. The new definition amounts to replacing the uncompensated demand functions of D.1 by compensated demand functions, i.e., the consumer's choice is that of minimizing the cost of achieving a given utility level. The relation between the two is the following: a demand vector which maximizes utility under a given budget constraint certainly minimizes the cost of achieving the resulting utility; but a demand vector which minimizes the cost of achieving some stated utility also maximizes utility without spending more if the amount spent is positive (but not in general if $M_h = 0$).

D.2. The price vector, p^*, utility allocation, (u_h^*), and allocation (x_h^*, y_f^*) is a <u>compensated equilibrium</u> if,

(a) $\underset{\sim}{p}^* \geq 0$ but $\underset{\sim}{p}^* \neq 0$;

(b) $\sum_h \underset{\sim}{x}_h^* \leq \sum_h \bar{\underset{\sim}{x}}_h + \sum_f \underset{\sim}{y}_f^*$;

(c) $\underset{\sim}{y}_f^*$ maximizes $\underset{\sim}{p}^* y_f$ subject to $y_f \in Y_f$;

(d) $\underset{\sim}{x}_h^*$ minimizes $\underset{\sim}{p}^* \underset{\sim}{x}_h$ subject to $U_h(\underset{\sim}{x}_h) \geq u_h^*$;

(e) $\underset{\sim}{p}^* \underset{\sim}{x}_h^* = M_h^*$.

From the previous remarks, we can note,

<u>Lemma 1</u>. If $(\underset{\sim}{p}^*, \underset{\sim}{x}_h^*, \underset{\sim}{y}_f^*)$ constitute a competitive equilibrium and $u_h^* = U_h(\underset{\sim}{x}_h^*)$, all h, then $(\underset{\sim}{p}^*, u_h^*, \underset{\sim}{x}_h^*, \underset{\sim}{y}_f^*)$ constitute a compensated equilibrium. If $(\underset{\sim}{p}^*, u_h^*, \underset{\sim}{x}_h^*, \underset{\sim}{y}_f^*)$ constitute a compensated equilibrium and if $M_h^* > 0$, all h, then $(\underset{\sim}{p}^*, u_h^*, \underset{\sim}{x}_h^*\ \underset{\sim}{y}_f^*)$ constitute a competitive equilibrium.

Hence, to establish the existence of a competitive equilibrium it suffices to establish the existence of a compensated equilibrium such that $M_h^* > 0$, all h. Two of the conditions stated above together are sufficient to insure this.

<u>Lemma 2</u>. If assumptions III and VII hold, then $M_h^* > 0$, all h, at a compensated equilibrium, so that it is also a competitive equilibrium.

The argument runs roughly as follows: At a compensated equilibrium, firms are maximizing profits, by D.2(c). Since the firm can always shut down, by assumption I, equilibrium profits must be non-negative, so that, from (2),

$$M_h^* \geq 0, \text{ all h.} \tag{3}$$

Also, from profit maximization,

$$p^* y_f^* \geq p^* \bar{y}_f,$$

where $\bar{y}_f$ is the output-input vector for firm f referred to in assumption III. Sum over firms f and add $p^* \bar{x}$; from (2),

$$\sum_h M_h^* = \sum_h (p^* \bar{x}_h) + \sum_f \sum_h d_{hf} (p^* y_f^*)$$

$$= p^* \bar{x} + \sum_f (p^* y_f^*) \geq p^* (\bar{x} + \sum_f \bar{y}_f),$$

since $\sum_h d_{hf} = 1$, by (1). But from III, all the components of $\bar{x} + \sum_f \bar{y}_f$ are positive, while from D.2(a), all components of p^* are non-negative and at least one positive. Hence,

$$\sum_h M_h^* > 0,$$

which implies,

$$M_h^* > 0 \text{ for some } h = h'', \text{ say.} \tag{4}$$

Suppose household h' is resource-related to household h". Then the assets held by h' are valuable to h", in the sense that his utility could be made to increase if the h'-assets increased; also h" has an effective demand, since it has a positive income. It is then reasonable to assert and can be provided rigorously that at least one of the h'-assets must command a positive price. But this means, from (2), that $M_h^* > 0$ for $h = h'$. In turn that implies that $M_h^* > 0$ for any h resource-related to h'. Continuing in this way leads to the conclusion that $M_h^* > 0$ for any h indirectly resource-related to h"; but by VII, that includes every household, so that Lemma 2 holds.

We can therefore confine attention to the existence of a compensated equilibrium. One possible way of proceeding is to make use of the familiar relations between the competitive price system and Pareto efficiency. To simplify the discussion, we use some notation: an allocation (x_h, y_f) will be abbreviated to w. The set of all possible allocations will be denoted by W; the set of __feasible allocations__, to be denoted by $\hat{W}$, are those for which,

$$\sum_h x_h \leq \bar{x} + \sum_f y_f.$$

Clearly, if $w = (x_h, y_f)$ is a feasible allocation, then (y_f) is a feasible production allocation, since $x_h \geq 0$, all h. As noted earlier, the set of feasible production allocations is closed, bounded, and convex; from this, it is immediate that,

$\hat{W}$, the set of feasible allocation, is closed, bounded, and convex. (5)

Any feasible allocation $w = (x_h, y_f)$ determines a utility level, $u_h = U_h(x_h)$ for each household. The numbers (u_h) taken as a vector will be termed a __utility allocation__, denoted by $\underset{\sim}{u}$. We define a <u>Pareto efficient utility allocation</u> in a slight variation of the usual manner:

D.3. The utility allocation $\underset{\sim}{u}$ is Pareto efficient if there is no other (feasible) utility allocation, $\underset{\sim}{u}'$, such that $u_h' > u_h$ for all h.

By the basic theorem of welfare economics, there is associated with each Pareto efficient utility allocation, $\underset{\sim}{u}^o$, a price vector, p^o, and a feasible allocation, $\omega^o = (\underset{\sim}{x}_h^o, \underset{\sim}{y}_f^o)$ such that,

(a) $\underset{\sim}{p}^o \geq 0$, $\underset{\sim}{p}^o \neq 0$;

(b) x_h^o minimizes the cost, $\underset{\sim}{p}^o\, x_h$, of achieving a utility level, $U_h(x_h)$, at least equal to u_h^o;

(c) $\underset{\sim}{y}_f^o$ maximizes profits, $\underset{\sim}{p}^o\, \underset{\sim}{y}_f$, among production vectors in Y_f;

(d) aggregate expenditures equals aggregate income, i.e.,

$$\sum_h p^o x_h^o = \sum_h p^o \bar{x}_h + \sum_f p^o y_f^o.$$

Actually, when there are constant returns to scale and/or production possibility sets which formed from finitely many basic activities (the linear programming model), it is not difficult to see that the price vectors and allocations realizing an efficient utility allocation may not be unique. Thus we can state in formal language,

<u>Lemma 3</u>. For every Pareto efficient utility allocation,

u^o, there is a set of prices, $P(u^o)$, and a set of feasible allocations, $\hat{W}(u^o)$, such that (a-d) above hold for every p^o in $P(u^o)$ and w in $\hat{W}(u^o)$.

Notice that every price vector in $P(u^o)$ supports every allocation in $\hat{W}(u^o)$. It is not hard to observe from this that the sets $P(u^o)$ and $\hat{W}(u^o)$ are convex sets.

The lemma associates with each utility a vector a set of prices (and similarly a set of allocations). This relation generalizes the usual concept of a function, which associates a number or vector with each vector. A relation which associates a set to each vector is sometimes termed a set-valued function, sometimes a correspondence; we follow Debreu [1959, Sections 1.3, 1.8] in using the latter term here. The concept of continuity is important in dealing with ordinary functions; we will need a generalization of it here.

D.4. A correspondence, which associates the set $\Phi(x)$ to the vector x, is said to be upper semi-continuous (u.s.c.) if, given a sequence $\{x^\nu\}$ approaching x^o and a sequence $\{y^\nu\}$ approaching y^o, where for each ν, y^ν is an element of the set $\Phi(x^\nu)$ associated with x^ν, then x^o belongs to $\Phi(x^o)$.

In figure 1 is illustrated the graph of an upper semi-continuous correspondence where, in addition, $\Phi(x)$ is a convex set (possibly consisting of a single point) for each x.

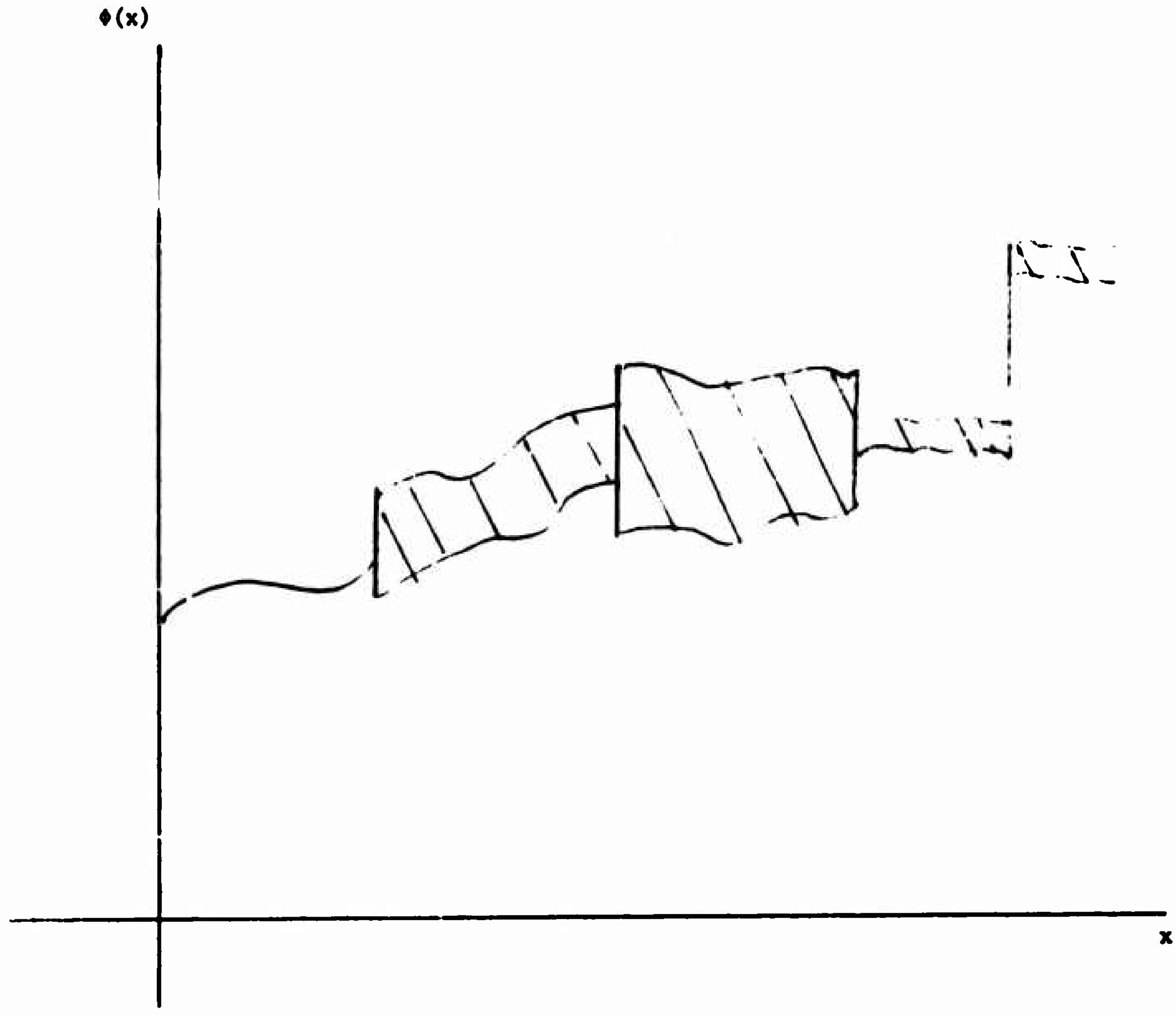


Figure 1

By straightforward if slightly tedious arguments, it can be shown that,

<u>Lemma 4</u>. The correspondences, $P(u)$ and $\hat{W}(\underset{\sim}{u})$, defined in Lemma 3, are u.s.c. and convex for each u.

Let us go back for a minute to assumption V; this guarantees the existence of a minimal consumption vector, $\bar{\bar{x}}_h$, available to household h at any set of prices whatever. In discussing competitive equilibrium, then, we can confine ourselves to consumption vectors which yield at least as much utility as $\bar{\bar{x}}_h$. We can assume, with no loss of generality, that,

$$U_h(\bar{\bar{x}}_h) = 0, \tag{6}$$

and confine our attention to utility allocations which yield each household utility at least equal to 0. Let, therefore,

U be the set of non-negative Pareto efficient utility allocations. (7)

For any feasible allocation, w in $\hat{W}$, and any price vector $\underset{\sim}{p}$, the expenditures of each household, $\underset{\sim}{p}\, x_h$, and its income, M_h, as given by (2), are defined, and hence so is its <u>budget surplus</u>,

$$s_h(p,w) = p\, x_h - M_h. \tag{8}$$

If we start with an arbitrary price vector and feasible allocation, we will "correct" them by imposing a penalty for violating the budget constraint, i.e., for a negative value of s_h.

This is done as follows: to each given price vector, $\underset{\smile}{p}$, and feasible allocation, ω, we associate the set of all non-negative utility allocations which yield 0 utility for those households with budget deficits. (The correspondence thus defined might be said to punish the improvident while being neutral with regard to others.) Formally, define,

$U(p, \omega)$ is the set of all non-negative Pareto efficient utility allocations, $\underset{\sim}{u}$, such that $u_h = 0$ for all households, h, for which $s_h(\underset{\smile}{p}, \omega) < 0$. (9)

To show the existence of a compensated equilibrium, we use the method of fixed points. That is, we start with a set of values for interesting economic magnitudes (in the present application, prices, utilities, and allocations). To each vector in the set we associate a vector in the set, or more generally, a set of vectors which is a subset of the original set. In the terminology we have introduced, we have a correspondence which maps the elements of some set into subsets of that set. Then under certain continuity hypotheses we find that there is at least one point of the set which belongs to the subset into which it is mapped by the correspondence. If the correspondence has been suitably constructed, then it can be shown that its fixed point is in fact the desired compensated equilibrium. The fixed point theorem used here is that due to the mathematician Kakutani [1951]. Kakutani's theorem is in turn derived from the fixed point theorem of Brouwer. An excellent reasonably elementary exposition of the proof of Brouwer's theorem is to be found in Tompkins [1964]; simple self-contained proofs of both theorems are given in Burger [1963, Appendix].

Lemma 5. (Kakutani's Fixed Point Theorem) Let S be a closed, bounded and convex set and $\Phi(x)$ a correspondence defined for $\underset{\sim}{x}$ in S and u.s.c. such that, for each $\underset{\sim}{x}$, $\Phi(\underset{\sim}{x})$ is non-null, and convex. Then there is some $\underset{\sim}{x}^{o}$ such that $\underset{\sim}{x}^{o}$ belongs to $\Phi(\underset{\sim}{x}^{o})$.

In our application, the elements of S will be triples $(\underset{\sim}{p}, \underset{\sim}{u}, \omega)$ consisting of price vectors, $\underset{\sim}{p}$, non-negative Pareto efficient utility allocations, $\underset{\sim}{u}$ in U (see (7)), and feasible allocations, ω in $\hat{\Omega}$. The price vectors are assumed to be non-negative and have at least one positive component. Since multiplication of all prices by a positive constant has no economic significance, we can normalize the prices in some convenient way; we choose to make the sum of all prices equal to one. Then define the range of prices to be the set satisfying the conditions,

P is the set of price vectors $\underset{\sim}{p}$, with $\underset{\sim}{p} \geq \underset{\sim}{0}$. and $\sum_i p_i = 1$. (10)

The domain S is then the set of all triples $(\underset{\sim}{p}, \underset{\sim}{u}, \omega)$ with $\underset{\sim}{p}$ in P, $\underset{\sim}{u}$ in U, and ω in $\hat{\Omega}$; each of the three components varies independently over its range. The set of all such triples is most conveniently denoted by,

$$P \times U \times \hat{\Omega},$$

and is referred to as the Cartesian product of the three sets.

More generally, given m sets, $X_1,\ldots,X_m$, their Cartesian product,

$$X_1 \times X_2 \times \ldots \times X_m,$$

is the set of all multiples of vectors, $(x_1,\ldots,x_m)$, such that x_1 belongs to X_1, x_2 to $X_2,\ldots,x_m$ to X_m.

To each point (p, u, w) in $P \times U \times \hat{W}$, we associate a set which is the Cartesian product.

$$P(u) \times U(p,w) \times \hat{W}\ (\underset{\sim}{u}). \tag{11}$$

It is easy to see that P is a closed bounded set; since feasible allocations are bounded, by (5), it also follows that U is closed and bounded. Hence, the domain $P \times U \times \hat{W}$ is closed and bounded. The set P is convex, and the same is true of $\hat{W}$ by (5). It is not necessarily true that U is convex, however; it is after all simply the utility-possibility surface, and its shape is indeed dependent upon the choice of the utility indicator for each household, a choice which depends upon an arbitrary monotone transformation. For the moment, however, pretend that U is convex.

As asserted in Lemma 4, $P(u)$ and $\hat{W}(u)$ are u.s.c. and convex for each; they are non-null by Lemma 3. It is easy to verify that $U(\underset{\sim}{p}, w)$ is non-null for each $\underset{\sim}{p}$, w, and that it is an u.s.c. correspondence. Pretend again that it is also convex. Then the Cartesian product, (11), can easily be verified to be u.s.c. in the variables, $\underset{\sim}{p}$, $\underset{\sim}{u}$, w, and to be non-null and convex for each set of values of the variables. Then Kakutani's theorem, Lemma 5, assures that there is a fixed point, i.e., a triple, $(\underset{\sim}{p}^*, \underset{\sim}{u}^*, w^*)$ such that,

$$(\underset{\sim}{p}^*, \underset{\sim}{u}^*, w^*) \text{ belongs to } P(\underset{\sim}{u}^*) \times U(\underset{\sim}{p}^*, w^*) \times \hat{W}(\underset{\sim}{u}^*).$$

By definition of a Cartesian product, this is equivalant to the three statements,

$$\underset{\sim}{p}^* \text{ belongs to } P(\underset{\sim}{u}^*), \tag{12}$$

$$\underset{\sim}{u}^* \text{ belongs to } U(\underset{\sim}{p}^*, w^*), \tag{13}$$

$$w^* \text{ belongs to } \hat{W}(\underset{\sim}{u}^*). \tag{14}$$

From (12) and (14), we can apply Lemma 3. Statements (a-c) of Lemma 3 together with the definition of $\hat{W}(\underset{\sim}{u}^*)$ as containing only feasible allocations yield immediately statements (a-d) of D.2. It remains only to verify D.2(e). In view of (8), this is equivalent to showing that,

$$s_h(\underset{\sim}{p}^*, w^*) = 0, \text{ all } h. \tag{15}$$

On the other hand, statement (d) of Lemma 3, is equivalent to,

$$\sum_h s_h(\underset{\sim}{p}^*, w^*) = 0;$$

hence, to prove (15) it suffices to show that,

$$s_h(p^*, \omega^*) \geq 0, \text{ all } h, \tag{16}$$

for, if a sum of non-negative quantities is zero, each must be zero. Suppose then that (16) is false,

$s_h(p^*, \omega^*) < 0$, some h.

Then (13) and (9) together imply that $u_h^* = 0$ for any such h. But D.2(d) has already been demonstrated, i.e., at a compensated equilibrium each household is attaining its utility at minimum cost. By our convention (6), $u_h^* = 0$ can always be attained by choosing the consumption vector $\bar{\bar{x}}_h$, and this vector, by V, can always be obtained without a budget deficit, so that (16) holds and therefore (15); condition (e) of D.2 is now verified and the demonstration of the existence of compenstated equilibrium completed. From Lemma 2, then, the existence of a competitive equilibrium is now demonstrated.

We left one loose end; the application of Kakutani's theorem seems to require the convexity of U, the set of non-negative Pareto efficient allocations, and of $U(p, \omega)$, as defined in (9). We can relate U and $U(p, \omega)$, however, to convex sets in a straightforward way; the process is illustrated in figure 2. Let V be the set of vectors $\underline{v}$, with as many components as households, such that,

V is the set of vectors v for which $v \geq 0$, $\sum_h v_h = 1$.

It is obvious that $\underline{O}$ is not a Pareto efficient utility allocation. Hence, a line drawn from the origin to an element of U intersects

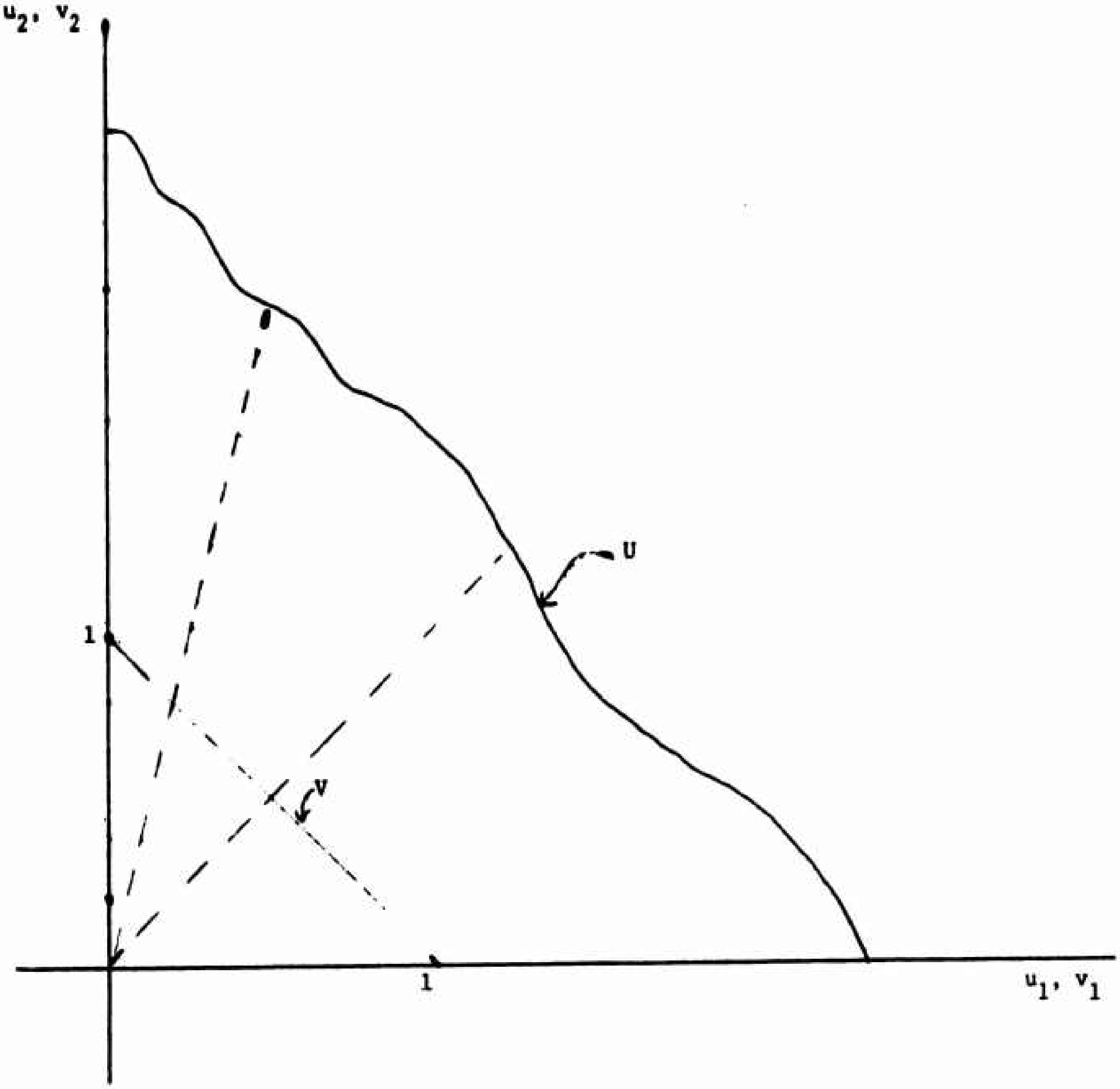


Figure 2

V once and only once and can be used to associate a point of V to it. Therefore, selecting an element of U is equivalent to selecting an element of the convex set, V. Further, a member of U for which $u_h = 0$ is associated in this way with a point for which $v_h = 0$. By (9), $U(p, w)$ consists precisely of points of U for which $u_h = 0$ for certain h; it is therefore associated with a set, $V(p, w)$, which consists of those points of V for which $v_h = 0$ for the same h. (In figure 2, if $U(p, w)$ is defined by the condition $u_1 = 0$, then it consists of the one point of U on the u_2-axis and is associated with the unique point of V for which $v_1 = 0$.) Then $V(p, w)$ is a convex set.

If then we replace U and $U(p, w)$ by V and $V(p, w)$ in the above mapping, all the conditions of Kakutani's theorem are strictly fulfilled.

3. The Firm as Price-Maker: Equilibrium under Monopolistic Competition

We now assume that there are some firms in the economy which are capable of exercising monopolistic or monopsonistic power over certain markets. We assume however the absence of interaction among the monopolistic firms. Each firm takes the current prices of products not under its control as given and perceives a demand (or supply) function, which may or may not be correct. The perception is made on the basis of observed prices and allocation. It is assumed that, at least at equilibrium, the demand functions are correct at the observed point, though not necessarily elsewhere. I.e., for the quantities actually produced, the firm correctly perceives the prices which will clear the markets. However, it is not necessarily assumed that the monopolistic firms correctly perceive the elasticities of demand at the equilibrium point.

A model with these properties was developed in a brilliant paper by Negishi [1951], and an existence theorem proved for it, the only previous work of this type known to me. The assumptions made here are much weaker than those of Negishi; comparisons between the present model and earlier models of monopolistic competition, including Negishi's, are made at the end of this section.

The production psosibility sets for monopolistic firms need not be convex; indeed, it is presumably the non-convexity (in particular, the increasing returns to scale) which is the reason for the existence of monopoly. However, it is assumed that the prices charged by monopolistic firms are continuous functions of other prices and other production and consumption decisions. If we assume in the usual way that monopolists are maximizing profits according to their perceived demand curves, then this assumption amounts to saying that the perceived marginal revenue curves fall more sharply than marginal cost curves.

Though we weaken the convexity assumptions on the production possibility sets of the monopolists, we will still need to make some hypotheses which will insure that the set of feasible production allocations satisfies some reasonable conditions, specifically that it is bounded if resources are bounded and that it is a set which does not break up into several parts or have holes in its middle. The second provision will be expressed more precisely by requiring that the set of those production possibility vectors for the monopolistic sector which are compatible with feasibility for the entire production sector can be expressed as the image of a closed bounded set under a continuous mapping.

The assumptions on the competitive sector will remain those made before.

There are then two kinds of firms, competitive and monopolistic. A <u>subscript</u> C <u>or</u> M will indicate a vector of <u>all</u> commodities which is possible for a competitive firm or for the competitive sector as a whole. Thus, Y_{Cf} is the production possibility set for competitive firm f, Y_{Mg} for monopolistic firm g. The production possibility set for the competitive sector as a whole is,

$$Y_C = \sum_f Y_{Cf},$$

and similarly, $Y_M = \sum_g Y_{Mg}$. The elements of these sets are represented by lower case bold face $\mathbf{y}$ with the appropriate subscripts. A monopolized commodity will of course not be the output of any vector in Y_C but it may be an input. Also, we use the term, "monopolized," to include, "monopsonized."

I. Assumption I of section 2 holds for the sets Y_{Cf}.

II. $\mathbf{0}$ belongs to Y_{Mg} and Y_{Mg} is closed, for each g.

It is possible to make assumptions parellel to II of section 2 (impossibility of getting something for nothing) to include the monopolistic firms. To avoid complications, we will simply assume the implication we there drew from this assumption. By a <u>production allocation</u> $(\mathbf{y}_{Cf}, \mathbf{y}_{Mg})$ we mean a specification of the production vector for each firm, competitive or monopolistic. A production allocation is <u>feasible</u> if,

$$\sum_f \mathbf{y}_{Cf} + \sum_g \mathbf{y}_{Mg} + \bar{\mathbf{x}} \geq 0. \tag{1}$$

Then, we assume,

III. The set of feasible production allocations is closed and bounded. (2)

An <u>allocation</u> is, as before, a consumption allocation and a production allocation, i.e, a complete specification $(\mathbf{x}_h, \mathbf{y}_{Cf}, \mathbf{y}_{Mg})$. Let $\mathcal{W}$ be the set of all possible allocations, $\hat{\mathcal{W}}$ is the set of feasible allocations, i.e., those for which, $\sum_h \mathbf{x}_h \leq \bar{\mathbf{x}} + \sum_f \mathbf{y}_{Cf} + \sum_g \mathbf{y}_{Mg}$. (3)

If we continue to assume, as we will, that consumption vectors, x_h, are always non-negative, then from the definition (3) and assumption III, it follows immediately that,

$\hat{W}$ is a closed bounded set. (4)

We introduce the concept of feasibility separately for the competitive sector (including households) and the monopolistic sector. An allocation in the competitive sector, w_C = (x_h, y_{Cf}) is feasible if there exists a monopolistic production allocation (not excluding 0) such that the entire allocation (x_h, y_{Cf}, y_{Mg}) is feasible. Similarly, a monopolistic production allocation, $y_M = (y_{Mg})$, is feasible if there exists an allocation in the competitive sector, w_C, such that the entire allocation (w_C, y_M) is feasible. Let,

$\hat{W}_C$ be the set of feasible allocations in the competitive sector, (5)

$\hat{Y}_M$ be the set of feasible monopolistic production allocations, (6)

Then (4) immediately implies,

$\hat{W}_C$ is closed and bounded. (7)

$\hat{Y}_M$ is closed and bounded (8)

We now make a basic assumption on the structure of the monopolistic production possibility sets which amounts to saying that the extent of increasing returns there is not too great relative to the resources that the competitive sector would be capable of supplying. Let,

$$z_C = \sum_h x_h - \bar{x} - \sum_f y_{Cf},$$

be any possible excess demand vector of the competitive sector, and Z_C be the set of all such z_C. In effect, $- z_C$ is the vector of amounts made available to the monopolistic sector by the competitive sector; in general, z_C may have some positive components among the monopolized goods, which correspond to demands by the competitive sector on the monopolistic sector. For simplicity, suppose there is only one monopolistic firm and let Y_M be its production possibility set. Then from the definition of feasibility (3), y_M is feasible if and only if,

$$y_M \text{ belongs to } Y_M,\ y_M \geq z_C \text{ for some } z_C \text{ in } Z_C. \tag{9}$$

For only one monopolistic firm, a monopolistic production allocation is simply the production vector for that firm, so that (9) characterizes the set of feasible monopolistic production allocations, $\hat{Y}_M$.

For simplicity, assume that there is free disposal in both the monopolistic and the competitive sectors. Then (9) states that $\hat{Y}_M$ is simply the intersection of the two sets, Y_M, the monopolist's production possibility set, and Z_C, the feasible excess demand vectors of the competitive sector. The set Z_C is convex by the assumptions made, but Y_M is not in general. The relation among these sets is illustrated in figure 3. If the competitive sector is large relative to the monopolistic, then the set Z_C will tend to be shifted to the left. The intersection, $\hat{y}_M$, will be "fat." It will then follow that if we inscribe a bounded closed convex set A, e.g., a sphere, as illustrated, every point of $\hat{y}_M$ can be projected into some element of the sphere (including its interior) in a continuous way and conversely every point in $\hat{y}_M$ is the projection of some point in the sphere. We will make this an assumption, although it can be derived from more primitive assumptions.

IV. There exists a continuous function, say $y_M(a)$, which maps a closed bounded convex set, A, into all points of $\hat{y}_M$.

It is worth illustrating that if Z_C is not large relative to Y_M, then IV need not hold: see figure 4, which is the same as figure 3 except for the location of the boundary of Z_C. Now $\hat{y}_M$ breaks up into two parts, and certainly cannot be the continuous projection of any one convex set. It is now certainly conceivable that no equilibrium will exist (though no example has been constructed); from an initial allocation corresponding to one area, the monopolist might always be motivated to choose a price which moves demand into the other

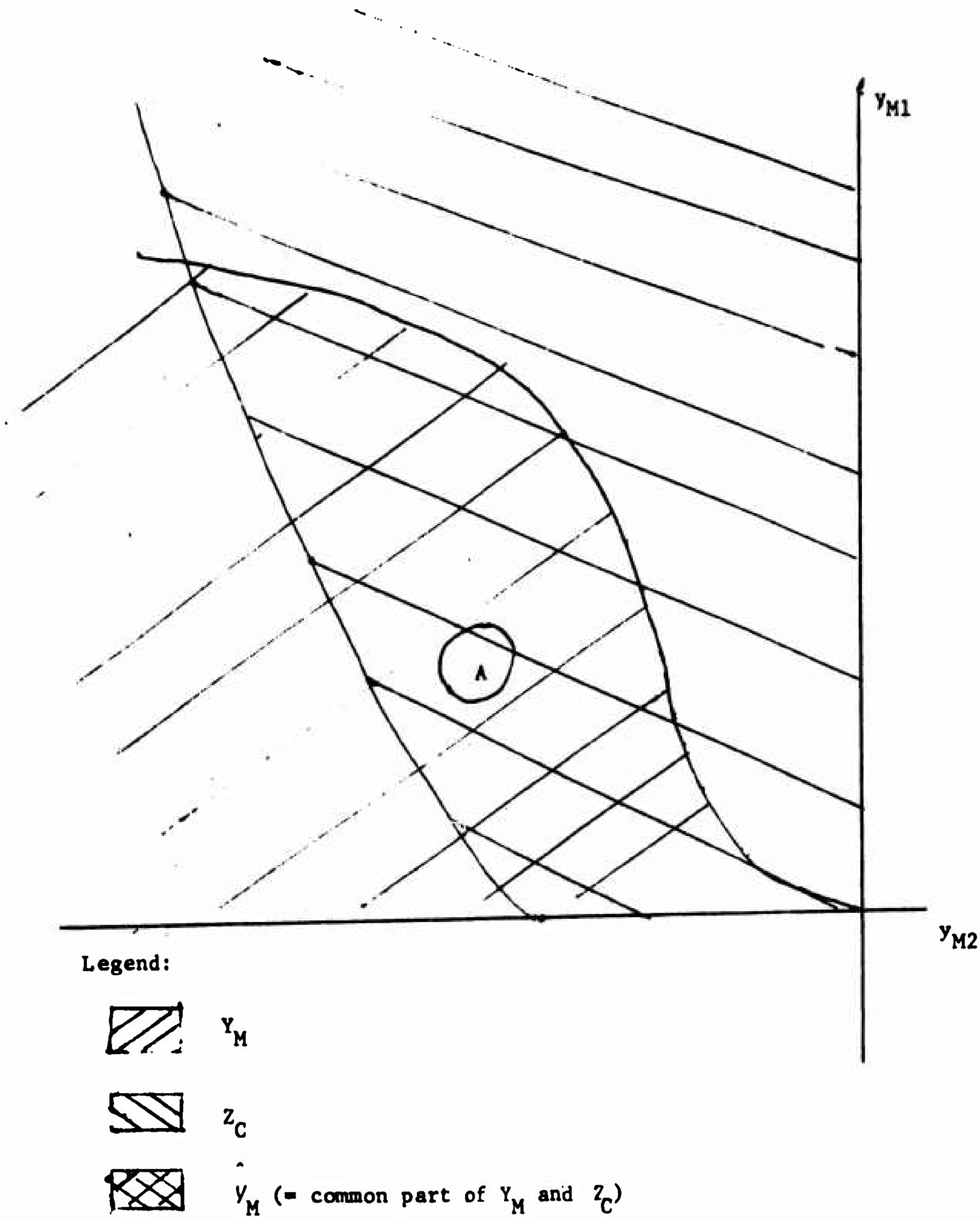
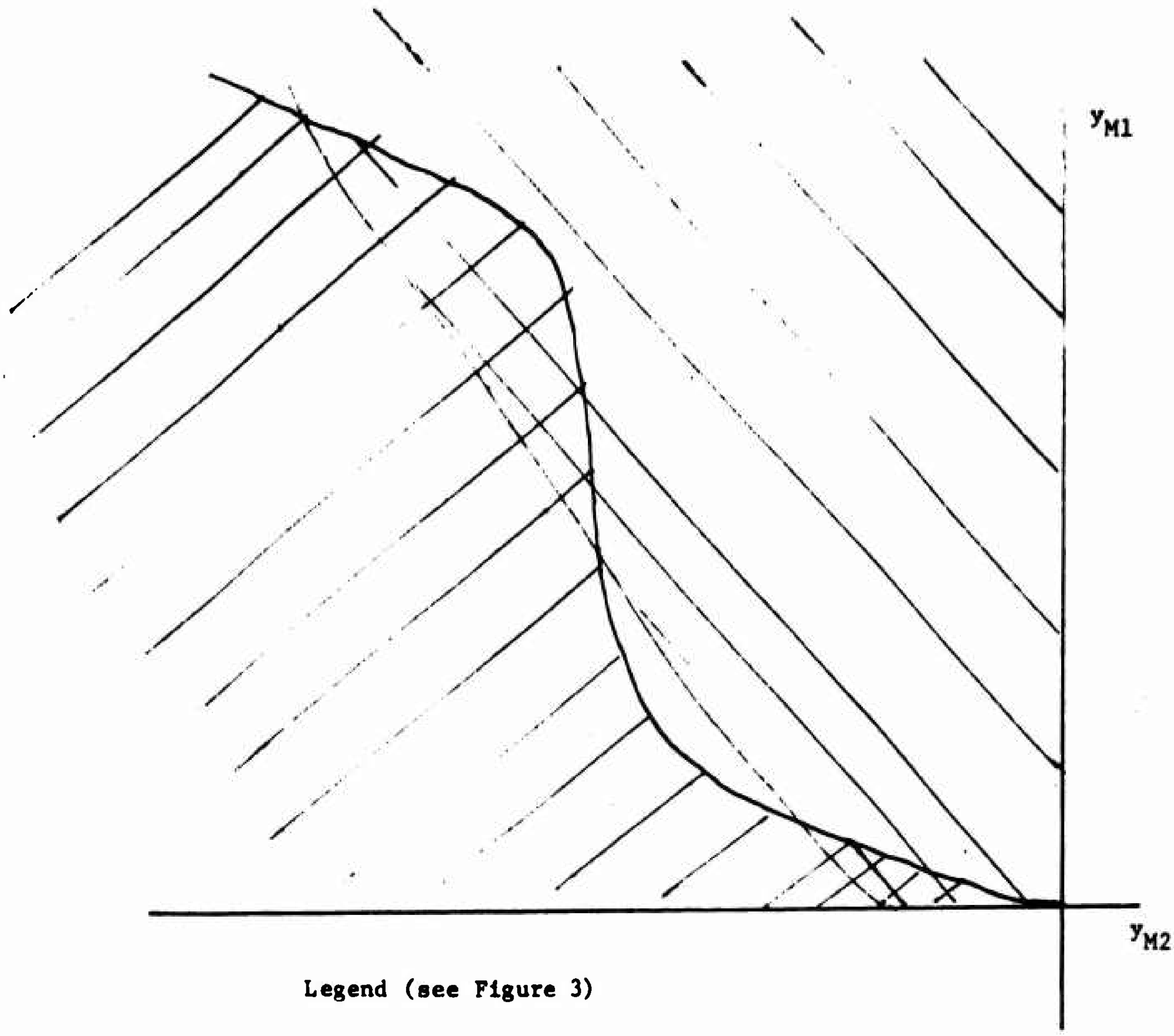


Figure 3



Legend (see Figure 3)

Figure 4

area. At the very least, the weak assumptions we will make on the monopolist's behavior will not suffice to exclude this possibility.

We now turn to the behavior of the monopolist. As a matter of notation, we will use a superscript M to denote those commodities which are monopolized or monopsonized, the superscript C for the remaining new commodities. Thus, $\underset{\sim}{p}^M$ is the vector of prices for the monopolized commodities alone; similarly, if P is a set of prices, P_C is the set of prices for competitive goods which is obtained by deleting from each $\underset{\sim}{p} \in P$ the components corresponding to monopolized goods.

At any given moment, the monopolists observe current prices and the current allocation and (individually) decide on their prices. We do not here derive their behavior from any hypothesis of profit - or utility - maximization, but take it for granted. The only conditions we impose are those indicated at the beginning of this section; monopolists' behavior is a continuous function of their observations, and the monopolist will change prices if his markets are not clearing. The second provision is somewhat complicated to state precisely in a general-equilibrium context. We take the following interpretation: suppose that the existing allocation is Pareto efficient within the competitive sector (i.e., taking the supplies and demands of monopolists as given) and that the relative prices of competitive goods correspond to a set of all prices which would support this allocation. Then if the entire set of prices (for monopolistic as well as competitive goods) will not support this allocation the monopolistic prices (or at least one of them) will change.

Before stating the assumption, we need some further notation. For any fixed monopolistic production allocation, y_M, there is a range of feasible allocations for the competitive sectors, provided $y_M \epsilon \hat{Y}_M$, namely those allocations, $w_C = (x, y_C)$ for which,

$$\sum_h x_h \leq \sum_f y_{Cf} + \bar{x} + \sum_g y_{Mg};$$

the productive activity of the monopolistic sector can be treated as a modification of the initial endowment from the viewpoint of the competitive sector.

We repeat the assumptions on consumer behavior in section 2 and add one mild condition.

V. Assumptions IV-VI of section 2 hold.

VI. No household is satiated in competitive goods alone, i.e., for every $x_h = (x_h^C, x_h^M)$, there exists $x_h^{C'}$ such that,

$$U_h(x_h^{C'}, x_h^M) > U_h(x_h) = U_h(x_h^C, x_h^M).$$

For fixed y_M, then, we can use the arguments of section 2 to note that there is a range of non-negative Pareto efficient utility allocations, $U(y_M)$, and, for each u in $U(y_M)$, a set of

Pareto efficient competitive allocations, to be denoted by,

$$\hat{\mathcal{W}}_C(u, \mathcal{Y}_M),$$

and a set of price vectors which sustain these allocations,

$$P_C(\underset{\sim}{u}, \mathcal{Y}_M).$$

(By "sustaining" the given allocations is meant that each firm is maximizing profits and each household is minimizing the cost of achieving the given utility levels at those prices.) The "Pareto efficiency" in question has to do only with allocation within the competitive sector for any given production behavior on the part of the monopolistic sector, and in no way implies the obviously false proposition that the allocation as a whole is Pareto efficient. Similarly, the price vectors in $P_C(\underset{\sim}{u}, \mathcal{Y}_M)$ only sustain the given allocation as far as the behavior of the competitive sector is concerned.

In accordance with our conventions about superscripts, the set, $P_C^C(\underset{\sim}{u}, \mathcal{Y}_M)$ is obtained by considering only those components of the price vectors in $P_C(\underset{\sim}{u}, \mathcal{Y}_M)$ which represent prices of competitive commodities. Then our assumption about the pricing behavior of the monopolistic sector reads,

VII. The prices charged by the monopolistic sector form a continuous function, $\underset{\sim}{p}^M(\underset{\sim}{p}, \mathcal{W})$, of prices and allocation. If $\underset{\sim}{p} = (\underset{\sim}{p}^C, \underset{\sim}{p}^M)$ and $\mathcal{W} = (x_h, y_{Cf}, y_{Mg})$ have the properties that, for some $\underset{\sim}{u}$ in $U(\mathcal{Y}_M)$,

$p^C = \lambda\, p^{C\prime}$ for some $\lambda \geqq 0$ and some $p^{C\prime}$ in $P^C_C(u, y_M)$,

$\omega_C = (x_h, y_{Cf})$ belongs to $\hat{\omega}_C(u, y_M)$,

but,

p does not belong to $P(u, y_M)$,

then,

$p^M(p, \omega) \neq p^M$.

It is further assumed that, if $\underset{\sim}{p}^M(\underset{\sim}{p}, \omega) = \underset{\sim}{p}^M$, then $\underset{\sim}{p}\, y_{Mg} \geqq 0$, all g, and that the sum of prices charged by monopolists does not exceed 1.

The next-to-last clause means that if monopolists are satisfied with their existing prices, they are not operating at a loss; the last clause means that monopolists, even though they make their decisions independently, will not, in total, demand more than is compatible with the normalization of prices.

Note that, from assumption VI, the prices of competitive commodities cannot all be zero if they sustain efficient allocation within the competitive sector.

For any y_M and any $\underset{\sim}{u}$ in $U(y_M)$, $\underset{\sim}{p}^C \neq 0$ for any $\underset{\sim}{p}$ in $P_C(\underset{\sim}{u}, y_M)$. (10)

Finally, we make an assumption corresponding to III of section 2, the ability of the economy to produce a positive amount of every good. We apply it however to the behavior of the competitive sector under the assumption that the monopolistic sector is not operating.

VIII. It is possible to choose $\bar{y}_{Cf}$ from Y_{Cf} for each f so that $\sum_f \bar{y}_{Cf} + \bar{x}$ is strictly positive in every component representing a competitive commodity and zero in every component representing a monopolized commodity.

In defining equilibrium for monopolistic competition, we must provide for the distribution of monopolistic profits to households. Hence, the income of the household is now given by,

$$M_h = p\, \bar{x}_h + \sum_f d^C_{hf}\, (p\, y_{Cf}) + \sum_g d^M_{hg}\, (p\, y_{Mg}). \tag{11}$$

where d^C_{hf} is the share of household h in the profits of competitive firm f and d^M_{hg} is the share of household h in the profits of monopolistic firm g, so that,

$$d^C_{hf} \geqq 0,\ d^M_{hg} \geqq 0,\ \sum_g d^C_{hf} = 1,\ \sum_g d^M_{hg} = 1,$$

and therefore,

$$\sum_h M_h = p\, \bar{x} + p\, (\sum_f y_{Cf}) + p\, (\sum_g y_{Mg}). \tag{12}$$

D.1. A price vector, p^*, and an allocation, $w^* = (x^*_h, y^*_{Cf}, y^*_{Mg})$, constitute a <u>monopolistic competitive equilibrium</u> if,

(a) $p^* \geq 0$ and $p^* \neq 0$;

(b) $\sum_h x^*_h \leq \sum_h \bar{x}_h + \sum_f y^*_{Cf} + \sum_g y^*_{Mg}$;

(c) y^*_{Cf} maximizes $p^* \, y_{Cf}$ subject to y_{Cf} in Y_{Cf};

(d) x^*_h maximizes $U_h(x_h)$ subject to $p^* \, x_h \leq M^*_h$

$$= p^*\bar{x}_h + \sum_f d^C_{hf} \; (p^* \, y^*_{Cf}) + \sum_g d^M_{hg} \; (p^* \, y^*_{Mg});$$

(e) $p^{M*} = p^M(p^*, w^*)$.

As in section 2, it is convenient to demonstrate first the existence of a closely related type of equilibrium.

D.2. A price vector, p^*, utility allocation, u^*, and an allocation $w^* = (x^*_h, y^*_{Cf}, y^*_{Mg})$ constitute a <u>compensated monopolistic equilibrium</u> if (a), (b), (c) and (e) of D.1. hold, and, in addition,

(d') x^*_h minimizes $p^* \, x^*_h$ subject to $U_h(x^*_h) \geq u^*_h$;

(f) $p^* \, x^*_h = M^*_h$.

We now construct the mapping used to prove the existence of compensated monopolistic equilibrium.

An allocation, w, specifies in particular, a monopolistic production allocation, y_M. Start, then, with an allocation, w, a utility allocation u which is Pareto efficient in the competitive sector for the given y_M, i.e., an element of $U(y_M)$, and a price vector, p. We form a set of price vectors associated with this triple as follows. The monopolistic components are assumed given by $p^M(p, w)$. For the given u

and y_M, the set $P_C(\underset{\sim}{u}, y_M)$ contains all vectors which would sustain that utility allocation in the competitive sector. For each $\underset{\sim}{p}$ in $P_C(\underset{\sim}{u}, y_M)$, consider the corresponding vector, $\underset{\sim}{p}^C$, containing just those components which represent competitive goods. By (10), $p^C \neq 0$. Hence, each such vector can be rescaled so that, with the given monopolistic components, the final price vector satisfies the normalization condition that the sum of all prices (monopolistic and competitive) equals one (the last clause of assumptions VII is also needed here). Formally,

$\tilde{P}(u, p, w)$ is the set of all vectors p such that $p^M = p^M(p, w)$, $p^C = \lambda\, p^{C'}$ for some $\lambda \geqq 0$ and some p in $P(u, y_M)$, and $\sum_i p_i = 1$. Here, y_M are the monopolistic production components of w, and u belongs to $U(y_M)$. (13)

Now define,

$$s_h(p, w) = p\, x_h - M_h(p, w), \tag{14}$$

where M_h is defined by (11), and, as in section 2, let,

$U(p, w)$ be the set of utility allocations in $U(y_M)$ such that $u_h = 0$ if $s_h(\underset{\sim}{p}, w) < 0$. (15)

Define P as before (see (9) of section 2). We start now with the quadruples (p, u, w_C, a), where p belongs to P, $\underset{\sim}{u}$ to $U(y_M)$, w_C to $\hat{W}_C$, and $\underset{\sim}{a}$ to A. The set A has been introduced in assumption IV. It will be recalled that w_C is an allocation in the competitive sector. From IV, $\underset{\sim}{a}$ define a monopolistic production allocation, $y_M(\underset{\sim}{a})$ in $\hat{Y}_M$, so that w_C and $\underset{\sim}{a}$, together, define an allocation, w. Then the quadruple (p, u, w_C, a) is mapped into the Cartesian production,

$$\tilde{P}(\underset{\sim}{u}, \underset{\sim}{p}, w) \times U(\underset{\sim}{p}, w) \times \hat{W}_C(\underset{\sim}{u}, y_M) \times \{a\},$$

where $\{a\}$ consists of the single point, a. Kakutani's theorem can then be applied to show the existence of a fixed point (p^*, u^*, w_C^*, a^*). Some difficult points in the proof are noted below. Let,

$$y_M^* = y_M(\underset{\sim}{a^*}), \; w^* = (w_C^*, y_M^*). \tag{16}$$

By construction, $\underset{\sim}{u}^*$ is Pareto efficient in the competitive sector for the given y_M^*. By definition of a fixed point,

$$p^* \text{ belongs to } \tilde{P}(u^*, p^*, w^*), \tag{17}$$

$$u^* \text{ belongs to } U(\underset{\sim}{p}^*, w^*), \tag{18}$$

$$w_C^* \text{ belongs to } \hat{W}_C(\underset{\sim}{u}^*, y_M^*). \tag{19}$$

From the definition of $P(u, p, w)$ in (13), (17) states that,

$$p^{C*} = \lambda\, p^{C'} \text{ for some } \lambda \geqq 0 \text{ and some } p^{C'} \text{ in } P_C^C(u^*, y_M^*),$$

and,

$$p^{M*} = p^M(p^*, w^*). \tag{20}$$

From assumption VII, (19) and (20) could not both hold if $\underset{\sim}{p}^*$ did not belong to $\dot{P}(\underset{\sim}{u}^*, y_M^*)$. Hence,

$$\underset{\sim}{p}^* \text{ belongs to } P(\underset{\sim}{u}^*, y_M^*). \tag{21}$$

Now (21) and (19) together show that the fixed point allocation and prices indeed define a Pareto efficient allocation within the competitive sector. From Lemma 3 of section 2, it follows, as in section 2, that conditions (a), (b), (c), and (d') of D.2 hold, while (20) asserts that (e) holds.

From (14) and (18), it follows, just as in section 2, that

$s_h(p^*, \omega^*) = 0$ for all h,

so that D.2(f) also holds. Thus, (p^*, u^*, ω^*) form a compensated monopolistic equilibrium.

The difficult points in applying Kakutani's theorem, referred to above, are the following: (1) The range of the variable $\underset{\sim}{u}$ now depends on y_M, since we assume $\underset{\sim}{u}$ belongs to $U(y_M)$; also, as in section 2, the range need not be a convex set. This can be met by extending the device used there; for each y_M, the set $U(y_M)$ can be mapped into a set V for which,

$v \geq 0, \sum_h v_h = 1.$

Since $\underset{\sim}{a}$ defines $y_M = y_M(a)$, $\underset{\sim}{a}$ and $\underset{\sim}{v}$ together define $\underset{\sim}{u}$ in $U(y_M)$, the correspondence used above can be considered as defined on,

$P \times V \times \hat{\omega}_C \times A,$

where P, V, and A are closed bounded convex sets. Similarly, any $\underset{\sim}{u}$ in $U(\underset{\sim}{p}, \omega)$ can be mapped into a member of V, with the property that $v_h = 0$ if and only if $u_h = 0$; it is not difficult to see that $V(\underset{\sim}{p}, \omega)$, so defined, is convex. (2) The set $\hat{\omega}_C$ is not necessarily convex; recall that is the set of allocations in the competitive sector which are feasible for <u>some</u> monopolistic production allocation; since the set of monopolistic production allocations is not in general convex, neither is $\hat{\omega}_C$ in general. However, the image set $\hat{\omega}_C(\underset{\sim}{u}, y_M)$ is always convex. Hence, all that is needed is to pick for the domain of definition a closed bounded convex set of competitive allocations containing $\hat{\omega}_C$.

From (10), (12), and assumption VIII, it follows just as in section 2, that,

$$\sum_h M_h^* > 0.$$

We can define resource-relatedness and indirect resource-relatedness as in section 2 with respect to the competitive sector alone, for any given feasible monopolistic production allocation (which affects the competitive sector as if it were a change in initial endowment).

IX. Every household is resource-related to every other for any given feasible monopolistic production allocation.

Then by the argument already given in section 2 a compensated monopolistic equilibrium is a monopolistic competitive equilibrium.

Theorem. Under assumptions I-IX, a monopolistic competitive equilibrium exists.

Remark 1. The model presented here is a formalization of Chamberlain's [1956, pp. 81-100; originally published in 1933] case of monopolistic competition with large numbers. As Triffin [1940] showed, the essential aspects of monopolistic competition appear as soon as one attempts to introduce some monopolies into a system of general competitive equilibrium. The only previous complete formalization is that of Negishi [1961]. Negishi assumed that each monopolist produced only one commodity and maximized profits according to a perceived demand curve which was a function of all prices and the allocation but was in particular linear in the price of the commodity. He saw the importance of a formulation of the type of VII here, that at equilibrium the monopolist's perceived demand curve should at least pass through the observed price-quantity point. In his formulation, which was originally suggested by Bushaw and Clower [1957, p. 181], the monopolist's price equalled the given one if at the given allocation, supply and demand were equal for that commodity. The assumption made here is considerably weaker, since it need only hold if the competitive sector is in equilibrium. Also Negishi restricted attention to the case where the monopolists have convex production possibility sets, a severe condition since under those circumstances the occurrence of monopoly is unlikely, as Negishi himself noted (p. 199, middle). He raised the possibility of more general assumptions, similar to IV.

Remark 2. No explicit mention has been made of product differentiation, a central theme of monopolistic competition theory. But note that the model admits the possibility that any monopolistic firm can produce a variety of goods. Suppose that all conceivable goods are included in the list of commodities; even what are usually regarded as varieties of the same good must be distinguished in this list if they are not perfect substitutes in both production and consumption. A monopolist will, in general, find it profitable to produce a number of varieties. The definition of a monopoly implies that, for some reason or another, two different monopolists produce non-overlapping sets of goods, but of course the goods produced by one monopolist may be quite close substitutes for those produced by another. The usual idea in product differentiation that a firm produces just one commodity is not a convenient assumption for general equilibrium analysis, but it is equally certainly not a good description of the real world.

Remark 3. The notion of free entry and with it the famous double-tangency solution of Chamberlain and Robinson [1933, pp. 93-4] have no role here either. The list of monopolists is assumed given, so that in effect there is a scarcity of the appropriate type of entrepreneurship, and there is no reason for profits to be wiped out. No doubt if there are several firms producing products which are close substitutes in consumption and have very similar production possibility sets otherwise, they should behave about the same way, and, if there are enough of them, it may well be that each is making very little in the way of profit. But the question then is the one raised originally by Kaldor [1935]: Would not the elasticity of demand to the individual firm be essentially zero, so that the situation is essentially one of perfect competition? It cannot be said that this question has been fully answered, since a more specific model defining close substitutes and their production possibilities has not yet been explicitly formulated.

Remark 4. An open and potentially important research area is the specification of conditions under which monopolistic behavior, as expressed in the function, $p^M(p, w)$, is in fact continuous. The formulation is very general; it is certainly compatible with utility-maximizing behavior (e.g., preference for size or particular kinds of expenditures or products) as well as profit-maximizing behavior. However, the assumption of continuity may nevertheless be strong; in effect, it denies the role of increasing returns as a barrier to entry. As the demand shifts upward, the firm might pass from zero output (i.e., a purely potential existence) to a minimum positive output. A zero output must be interpreted as a price decision at a level corresponding to zero demand; but if the demand curve is downward sloping, then entry at a positive level far removed from zero implies a discontinuous drop in price. The importance of this problem is not easy to assess. The situation can only arise if the (perceived) marginal revenue curve is, broadly speaking, flatter than the marginal cost curve, otherwise entry would be a continuous phenomenon; but then the demand curve must also be relatively flat and therefore the price discontinuity may be mild even if the output discontinuity is large. Also, if there were only a single monopolist who correctly perceived the excess demand correspondence of the competitive sector, he could choose his most preferred point, which would be then an equilibrium; the discontinuity of his behavior would be irrelevant. However, the problem may be important if his perceptions are accurate only at equilibrium or if there are several monopolists; the discontinuity in the behavior of any one affects the perceived demand functions for the others, though again, if the monopolists are relatively separated in markets and each relatively small on the scale of the economy, then the discontinuities involved may be unimportant.

Remark 5. It must always be remembered that monopolistic competition models of the type discussed here ignore the mutual recognition of power among firms, the oligopoly problem.

4. The Firm as Forecaster: The Existence of Temporary Equilibrium.

Hicks [1939, pp. 130-133] introduced the analysis of temporary equilibrium; a more recent methodological discussion is to be found in Hicks [1965, Chapter VI]. To interpret general equilibrium theory in the context of time, the formally simplest procedure is to regard commodities at different points of time as different commodities. But then we immediately encounter the somewhat unpleasant fact that the markets for most of these commodities do not exist. Since production and consumption both have important dynamic elements, individual agents replace the non-existent prices for future commodities by expectations (certain or probabilistic). Given these expectations equilibrium on current markets alone is arrived at (we here neglect the relatively few futures markets). There is however at least one current market in addition to those for the usual commodities, namely a market for bonds, to permit individuals to have planned expenditure patterns over time which differ from their income patterns.

We now understand that the components of the possible production and consumption vectors extend over several periods of time. For simplicity, we confine ourselves to two periods, present and future. We assume that the only commodities traded in currently are commodities of the current period plus bonds; a unit bond is a promise to pay one unit of the currency of account in the next period. Let the subscript, b, refer to bonds. We use here the notation, $\underset{\sim}{x}^1$, to refer to commodities of the current period, $\underset{\sim}{x}^2$ to those of the future period, and $\underset{\sim}{x}$ to be vector of commodities currently traded in by households; thus, $\underset{\sim}{x} = (\underset{\sim}{x}^1, x_b)$, not $x = (\underset{\sim}{x}^1, \underset{\sim}{x}^2)$. Similarly, for firms, $\underset{\sim}{y} = (\underset{\sim}{y}^1, y_b)$, where y_b is the supply of bonds issued by firms, and $\underset{\sim}{p} = (\underset{\sim}{p}^1, p_b)$ is the vector of prices on current markets.

Before going into details, we discuss the main difficulties in applying the methods of section 2 and the strategy for overcoming them. The modifications to be made to the definition of and assumptions on the production sets are straightforward except for one particular. Consider some one firm. Its production plan for the future has consequences in current markets because it affects the current value of the firm and the amount it will now borrow. But since there is no current market for the resources of the subsequent period, the availability of resources then cannot be used to argue that production plans are bounded. This creates obvious difficulties, which however are partly academic, since one could argue that a firm is "realistic enough" not to plan indefinitely large production. However, it is preferable to incorporate the argument from realism into our construction in a way more in the spirit of the perfectly competitive model. We do this by insisting that the price expectations of firms be "sensible". This is done in assumption II(b) below.

As already noted, the plans for future production must have consequences in current markets. We shall in fact assume that each firm offers in the current bond market a quantity of bonds equal to its expected profit in the future. This means that there is a "current" representation of the future plans, which in turn allows us to incorporate these in the framework of the model of section 2. This is made precise in (1) and D.2.

When we come to consumers, a number of special problems arise. First, we must decide what we mean by the initial endowment of bonds held by a household. We simply take it to equal its anticipated receipts of the future period; i.e., it represents the maximum the household believes it could repay. Note that the household's anticipated receipts may differ from what any other agent would expect them to be, given the household's plans - that is, we allow for differences in price expectations.

The differences in expectations however also means that different households will value any given firm differently. We assume that the actual current market value of any firm is equal to the highest value any agent places on it and suppose that the ownership of the firm will shift to the hands of that household or those households which value it most highly (D.5 and assumption IV). We therefore now treat d_{hf}, the share of household h in firm f, as a variable of the equilibrium. All this leads to modifications in the manner in which we must write the households' budget constraints (see (10) and (15)).

We must also insure that our assumptions about consumption possibility sets (IV and V of section 2) hold when reinterpreted in terms of current market. This is fairly straightforward; see assumption III, (16), and D.6.

Lastly there is the following problem. We know that the household utility depends on its future plans and in the existence proof of section 2, the utilities of the households play an important part. Our procedure of incorporating the expected future into arguments about the present is to use a "derived utility function" (D.7). This will be the maximum utility of the household, given its first period allocation, under an appropriate budget constraint. It will be obvious that this derived function can be treated as a function of current plans only, which is what we want.

We now proceed to detailed argument. Let us first consider the behavior of the firm. We take the viewpoint that the firm is an entity which, on its own, has expectations of future prices and maximizes profits in accordance with them. At the end of this section we will make some comments on this assumption.

We retain the assumptions on the production possibility sets of the individual firms with appropriate changes of notation, and add a hypothesis which embodies the possibility of abandoning a productive enterprise without loss.

D.1. The set of <u>possible two-period production vectors</u> for firm f is Y_f^{12}. An element is denoted by $y_f^{12} = (y_f^1, y_f^2)$, where y_f^1 are the components of y_f^{12} referring to the first period and y_f^2 those referring to the second period. We also refer to y_f^1 and y_f^2 as <u>first-period</u> and <u>second-period</u> production vectors respectively.

I. Assumption I of section 2 holds with y_f replaced by y_f^{12} and Y_f by Y_f^{12}. If $(y_f^1, y_f^2) \in Y_f^{12}$, then there exists $y_f^{2'} \geq 0$ such that $(y_f^1, y_f^{2'}) \in Y_f^{12}$.

The firm observes current prices, $\underset{\sim}{p} = (p^1, p_b)$, and is assumed to have subjectively certain expectations for prices in period 2, $\underset{\sim}{p}_f^2$; since there are no futures markets, different firms may have different expectations. A production plan, (y_f^1, y_f^2), yields net revenue $\underset{\sim}{p}^1 \underset{\sim}{y}_f^1$ in period 1 and is expected to yield net revenue $\underset{\sim}{p}_f^2 \underset{\sim}{y}_f^2$ in period 2. If bonds sell in period 1 at p_b, then a revenue of $\underset{\sim}{p}_f^2 \underset{\sim}{y}_f^2$ in period 2 is equivalent on perfect markets to a first-period income of $p_b \; (\underset{\sim}{p}_f^2 \underset{\sim}{y}_f^2)$. For simplicity, we will assume that the firm actually sells bonds to the extent of its anticipated second-period income, so that its offering of bonds is,

$$y_{fb} = \underset{\sim}{p}_f^2 \underset{\sim}{y}_f^2, \tag{1}$$

and its current receipts from a given production plan are,

$$\underset{\sim}{p}^1 \underset{\sim}{y}_f^1 + p_b \, (\underset{\sim}{p}_f^2 \underset{\sim}{y}_f^2). \tag{2}$$

The firm chooses its production plan so as to maximize (2) among all production plans $\underset{\sim}{y}_f^{12} \in Y_f^{12}$. Provisionally, we will assume that all price expectations are totally inelastic, i.e., that $\underset{\sim}{p}_f^2$ is a datum for the firm independent of current prices. Then (1) maps the elements of Y_f^{12} into a set Y_f of (n+1)-dimensional vectors; in effect, with fixed expectations, the firm's future possibilities amount to its ability to produce bonds for today's market.

D. 2. The set of possible current production vectors, $\underset{\sim}{y}_f$, for firm f is the set derived from Y_f^{12} by replacing the 2nd period components, y_f^2, by the single element obtained from them by (1), i.e.,

Y_f is the set of vectors $y_f = (y_f^1, y_{fb})$ such that $y_{fb} = p_f^2 y_f^2$ for some (y_f^1, y_f^2) in Y_f^{12}.

It is easy to verify from assumption I that,

assumption I of section 2 holds for Y_f under D.2. (3)

That is, Q belongs to Y_f (derived from $y_f^{12} = 0$); Y_f is closed, and Y_f is convex.

From D.2 and (2),

the firm maximizes $p\, y_f$. (4)

Suppose $p_b > 0$, and the firm has chosen a production plan, y_f^{12}, for which there will be negative receipts in the future, $p_f^2 y_f^2 < 0$. Then by the second half of I, it is possible to choose another plan with higher profits.

If $p_b > 0$, then $p_f^2 y_f^2 = y_{f,b} \geq 0$ at any profit-maximizing plan. (5)

We now make an assumption about the impossibility of production without inputs and about irreversibility which is somewhat stronger than that obtained by simply replacing y_f by y_f^{12} and Y_f by Y_f^{12} in assumption II of section 2. The reason the stronger assumption is needed is that future resource limitations do not directly restrain production, since there are no futures markets on which they appear. We do still have the constraints on first-period resources and in addition we will, in accordance with (5), restrict ourselves at certain stages in the argument to plans for which $p_f^2 y_f^2 \geq 0$, since only those will satisfy our equilibrium conditions.

II. (a) If $\sum_f y_f^1 \geq 0$, then $y_f^1 = 0$, all f. (b) The future returns to any production plan requiring no first-period inputs are bounded for any firm, i.e., $p_f^2 y_f^2$ is bounded as y_f^2 varies over all two-period production vectors $(0, y_f^2)$ in Y_f^{12} with no current inputs.

We will argue that this assumption is not unreasonable. First, it will have to be understood that any factor availabilities in period 1 as the result of earlier production (e.g., durable capital goods or maturing agricultural products) are to be included in the initial endowment of current flows, $\bar{x}^1$. Hence, the absence of net inputs means the absence of capital, labor, and current raw materials; it is reasonable then to conclude that no production takes place in period 1, i.e., $y_f^1 = 0$, all f. As far as (b) is concerned, if it were not true, then a firm could expect indefinitely large profits in the next period even if it were to shut down today. But then the firm would know that its price expectations are not consistent with any equilibrium and so it is reasonable to argue that it does not hold any such expectations. Thus (b) is really a weak requirement on the rationality of expectations.

It is convenient to define a two-period production allocation, (y_f^{12}) to be quasi-feasible if it is first-period feasible and if the second period components are not unprofitable to any firm (according to its own expectations, i.e., if it satisfies the conditions,)

$$\sum_f y_f^1 + \bar{x}^1 \geq 0, \quad p_f^2 y_f^2 \geq 0, \text{ each } f.$$

From assumption II, it is possible to prove, analogously to the corresponding discussion in section 2, that,

the set of quasi-feasible two-period production allocations is closed, bounded, and convex. (6)

In the theory of consumer behavior, we apply again the assumptions made earlier to the intertemporal consumption vectors.

D. 3. The set of possible two-period consumption vectors for household h is X_h^{12}, with elements, $x_h^{12} = (x_h^1, x_h^2)$, the components being referred to as the first-period and second-period possible consumption vectors, respectively.

III. Assumptions IV, V, and VI of section 2 hold under D.3 with x_h, $\bar{x}_h$, $\bar{\bar{x}}_h$, and X_h, replaced by x^{12}, $\bar{x}^{12}$, $\bar{\bar{x}}_h^{12}$, and X_h^{12}, respectively. We also assume that $U_h(x_h^1, x_h^2)$ is not satiated in x_h^2 for any x_h^1.

Like the firm, the household knows current prices, including that of bonds, and anticipates second-period prices, p_h^2. It plans purchases and sales for both periods. In each period, There is a budget constraint. The two constraints are linked through the purchase of bonds, which constitute an expense in period 1 and a source of purchasing power in period 2 (or vice versa, if the household is a net borrower in period 1). The household can be considered to have an initial endowment of bonds, $\bar{x}_{hb}$, which is precisely its anticipated volume of receipts in period 2. The net purchase of bonds in period 1 is then denoted by $x_{hb} - \bar{x}_{hb}$. Total expenditures for goods and bonds in period 1 are $\underset{\sim}{p}^1 \underset{\sim}{x}_h^1 + p_b(x_{hb} - \bar{x}_{hb})$, while planned expenditures in period 2 are $\underset{\sim}{p}_h^2 \underset{\sim}{x}_h^2$.

The purchasing power available in period 1 is the sum of the sale of endowment, $\underset{\sim}{p}^1 \bar{\underset{\sim}{x}}_h^1$, and receipts from firms in that period. The planned receipts in period 2 equals the planned sale of endowment, $\underset{\sim}{p}_h^2 \bar{\underset{\sim}{x}}_h^2$, plus receipts from firms in period 2, and this sum equals $\bar{x}_{hb}$, as remarked. The purchasing power planned to be available in period 2 is the repayment to the household of its net purchase of bonds, x_{hb} minus $\bar{x}_{hb}$, plus planned receipts, and is therefore simply x_{hb}.

There is a feature in this model not present in the static model or its intertemporal analogue with all futures markets. Since different households hold different expectations of future prices, they have different expectations of the profitability of any particular firm. Hence, a market for shares in firms will arise; the initial stockholders may value the firm less highly than some others, and therefore the stock of the firm should change hands.

After the firm has chosen its producion plan, y_f^{12}, household h values the plan according to current prices and its expectations of future prices. Let,

D. 4. The <u>capital value of firm f according to household h</u> is,

$$K_{hf}(\underset{\sim}{p}, \underset{\sim}{y}_f^{12}) = \underset{\sim}{p}^1 \underset{\sim}{y}_f^1 + p_b (\underset{\sim}{p}_h^2 \underset{\sim}{y}_f^2).$$

The value of the firm in the market is the highest value that any household gives to it.

D. 5. The <u>market capital value of firm f</u> is

$$K_f(\underset{\sim}{p}, \underset{\sim}{y}_f^{12}) = \max_h K_{hf}(\underset{\sim}{p}, \underset{\sim}{y}_f^{12}).$$

We will assume that, for each production plan for each firm, there is at least one household that values the plan at least as highly as the firm itself does; one might rationalize this by noting that the firm's manager is presumably himself the head of a household.

IV. The market capital value of a firm is at least equal to the maximum profits anticipated by the firm itself; in symbols,

$$K_f(\underset{\sim}{p}, \underset{\sim}{y}_f^{12}) \geq \underset{\sim}{p}\, \underset{\sim}{y}_f, \text{ for all } \underset{\sim}{p} \text{ and all } \underset{\sim}{y}_f^{12} \in Y_f^{12}.$$

From D.4. and D. 5,

$$K_f = \max_h [p^1 y_f^1 + p_b (p_h^2 y_f^2)] = p^1 y_f^1 + p_b \max_h (p_h^2 y_f^2),$$

since $p^1 y_f^1$ and p_b are independent of h. Let,

$$K_f^2(y_f^2) = \max_h (p_h^2 y_f^2). \tag{7}$$

If we recall that $p\, y_f = p^1 y_f^1 + p_b (p_f^2 y_f^2)$, then IV implies,

$$K_f - p\, y_f = p_b (K_f^2 - p_f^2 y_f^2) \geq 0. \tag{8}$$

Let $\bar{d}_{hf}$ be the share of firm f held initially by household h; we assume that it sells its shares at the market price and buys others, only however in those firms which it values at least as highly as any other household. We assume the absence of short sales. Let d_{hf} be its share of firm f after the stock market has operated. Its net receipts from sale less purchase of stocks (possibly negative, of course) are given by,

$$\sum_f (\bar{d}_{hf} - d_{hf}) K_f.$$

Also,

$$d_{hf} = 0 \text{ unless } K_{hf} = K_f. \tag{9}$$

It will be recalled that the current receipts of the firm are given by (2) or (4): it is assumed that they are all distributed among its new owners, so that household h receives,

$$\sum_f d_{hf}\,(p\,y_f).$$

Hence, the budget constraint for period 1 reads,

$$p^1\,x_h^1 + p_b\,(x_{hb} - \bar{x}_{hb}) \leq p^1\,\bar{x}^1 + \sum_f d_{hf}\,(p\,y_f) + \sum_f (\bar{d}_{hf} - d_{hf})K_f. \tag{10}$$

In period 2, the household is responsible for its share of the bonds issued by firm f, which total $p_f^2\,y_f^2$. But according to its expectations, the firm will receive $p_h^2\,y_f^2$. From (9), the household only invests in firms whose production plans it values at least as highly as anyone else, so that from (7) any firm for which $d_{hf} > 0$ will be expected by household h to have second-period receipts K_f^2. Hence, the anticipated total receipts from firms in period 2 by household h will be,

$$\sum_f d_{hf}\,(K_f^2 - p_f^2\,y_f^2).$$

From earlier remarks, then,

$$\bar{x}_{hb} = p_h^2\,\bar{x}_h^2 + \sum_f d_{hf}\,(K_f^2 - p_f^2\,y_f^2),\quad \bar{x}_h = (\bar{x}_h^1, \bar{x}_{hb}). \tag{11}$$

Then

$$\bar{x}_b = \sum_h \bar{x}_{hb} = \sum_h (p_h^2\,\bar{x}_h^2) + \sum_f (K_f^2 - p_f^2\,y_f^2). \tag{12}$$

Note that $\bar{x}_b$ is a function of the y_f^2's, the second-period production allocation. Note also that, from (8), the summation terms in (11) and (12) are non-negative.

Define now,

$$\bar{\bar{x}}_{hb} = p_h^2 \bar{\bar{x}}_h^2, \quad \bar{\bar{x}}_h = \bar{\bar{\underset{\sim}{x}}}_h = (\bar{\bar{\underset{\sim}{x}}}_h^1, \bar{\bar{x}}_{hb}). \tag{13}$$

By a slightly tedious but elementary calculation, it can easily be seen that the vector $\bar{\bar{\underset{\sim}{x}}}_h$, defined for current markets, in fact satisfies the conditions of assumption V of section 2 if assumption III above holds.

$$\bar{\underset{\sim}{x}}_h \geqq \bar{\bar{\underset{\sim}{x}}}_h; \text{ if } \quad \bar{x}_{hi} > 0, \text{ then } \bar{x}_{hi} > \bar{\bar{x}}_{hi}. \tag{14}$$

As already remarked, the budget constraint for period 2 is simply,

$$\underset{\sim}{p}_h^2 \underset{\sim}{x}_h^2 \leqq x_{hb}. \tag{15}$$

We therefore define,

D.6. The set of <u>current consumption vectors</u>, X_h, consists of all vectors, $\underset{\sim}{x}_h = (\underset{\sim}{x}_h^1, x_{hb})$ such that $x_{hb} \geqq p_h^2 \underset{\sim}{x}_h^2$ for some $(\underset{\sim}{x}_h^1, \underset{\sim}{x}_h^2)$ in X_h^{12}.

In other words, X_h is the set of current market vectors which, at the price expectations of the household, permit a possible two-period consumption vector.

From (15) and assumption III, $x_{hb} \geqq 0$, also $\bar{\underset{\sim}{x}}_h$ belongs to X_h. Assumption III, (14) and D.6 then assure us,

assumptions IV and V of section 2 hold with the new interpretations of x_h, $\bar{x}_h$, $\bar{\bar{x}}_h$, X_h (see (12), (14), and D.6, respectively). (16)

The maximization of $U_h(x^1_h, x^2_h)$ subject to the budget constraints (10) and (15) can be thought of as occurring in two stages. For any given $x_h = (x^1_h, x_{hb})$, we can maximize with respect to x^2_h subject to (15); the maximum is now a function of x^1_h and of x_{hb}, i.e., of x_h, with respect to which it can be maximized subject to (10).

D.7. <u>First period derived utility is</u>,

$$U^*_h(x_h) = \max U_h(x^1_h, x^2_h) \text{ subject to } p^2_h\, x^2_h \leq x_{hb}.$$

We do have to assume that the maximum in D.7. actually exists. The existence depends primarily on p^2_h, the household's anticipations of future prices. It will be assumed that the household is sufficiently realistic for this purpose; this is not an unreasonable assumption since the household would know, from the fact that a maximum does not exist, that the prices could not be equilibrium prices.

V. For given x^1_h, the function $U_h(x^1_h, x^2_h)$ assumes a maximum subject to the constraint $p^2_h\, x^2_h \leq x_{hb}$ for any x_{hb} permitting possible second period consumption, i.e., for any $x_h \in X_h$.

From III, it is easy to see that U^*_h is continuous. It is also true that it is semi-strictly quasi-concave and very easy to establish that U^*_h is locally non-satiated in x_h. By a suitable choice or origin, we can insure $U^*_h(\bar{\bar{x}}_h) = 0$.

$U^*_h(x_h)$ is continuous, semi-strictly quasi-concave, and admits no local satiation; $U_h^*(\bar{\bar{x}}_h) = 0$; U_h^* is strictly increasing in x_{hb} for any x_h^1. (17)

These properties, except for the last, are precisely those of U_h as assumed in VI of section 2.

The aim of the household, then, is to maximize U_h^* subject to (10) which can be written,

$$p\, x_h \leq M_h, \tag{18}$$

where,

$$M_h = \underset{\sim}{p}\, \bar{x}_h + \sum_f d_{hf}\, (\underset{\sim}{p}\, \underset{\sim}{y}_f) + \sum_f (\bar{d}_{hf} - d_{hf})\, K_f(p,\, y_f^1,\, y_f^2). \tag{19}$$

Another way of writing (19) will be useful. First, rewrite it slightly; then note that by our notation, $\underset{\sim}{p}\, \bar{x}_h = \underset{\sim}{p}^1\, \bar{x}_h^1 + p_b\, \bar{x}_{hb}$; then substitute from (11)

$$M_h = \underset{\sim}{p}\, \bar{x}_h + \sum_f \bar{d}_{hf}\, (\underset{\sim}{p}\, \underset{\sim}{y}_f) + \sum_f (\bar{d}_{hf} - d_{hf})\, (K_f - \underset{\sim}{p}\, \underset{\sim}{y}_f)$$

$$= \underset{\sim}{p}^1\, \bar{x}_h^1 + \sum_f \bar{d}_{hf}\, (\underset{\sim}{p}\, \underset{\sim}{y}_f) + p_b\, [p_h^2\, \bar{x}_h^2 + \sum_f \bar{d}_{hf}\, (K_f^2 - p_f^2\, y_f^2)]. \tag{20}$$

Recall that $K_f - \underset{\sim}{p}\, \underset{\sim}{y}_f = (K_f^2 - p_f^2\, y_f^2)$ by (8). One important implication of (20) is that the actual final share allocation does not affect the household budget constraints and therefore does not affect the equilibrium. The reason is that, since shares in firms are assumed to be sold to those who value them most highly at a price equal to that value, each potential buyer is in fact indifferent between making the purchase and investing in bonds, and none of his other behavior is affected by the choice.

We now have all the threads of the model in hand. Since equilibrium occurs only on current markets, the only relevant prices are those for current commodities and bonds. Basically, the model is very similar to that of static competitive equilibrium; the aim of the firm is to maximize $p y_f$ subject to $y_f \varepsilon Y_f$, according to (4) and D.2, while the consumer aims to maximize a (first-period derived) utility function subject to a budget constraint (18). The feasibility conditions for the current markets have the same form as before; demand for first-period commodities and for bonds shall not exceed supply, including the initial endowment of bonds as defined. However there are two complications: (1) the budget constraint, using the definition of M_h in (20), is somewhat different than that of Section 2 and more especially, contains variables, the y_f^2's, which are not the standard system; (2) by (12) one component of the social endowment vector, namely, $\bar{x}_b$, also depends on the y_f^2's.

Let us formally define competitive and compensated temporary equilibrium.

D.8. Competitive and compensated temporary equilibrium are defined as in section 2 (see D.1 and D.2 there) with the notation introduced in this section, except that (1) the variables y_f^2 must be consistent with intertemporal profit maximization, (2) the utility functions, U_h, are replaced by U_h^*, and (3) the budget equations now take the form,

$$p^* x_h^* = M_h^*$$

where M_h^* is given by (19) or (20) in terms of equilibrium magnitudes.

To prove the existence of compensated equilibrium, the previous mapping has to be only slightly modified; however, we here omit the details.

We have assumed to this point that all price expectations are totally inelastic; this assumption can easily be relaxed.

VI. For each household and firm, anticipated second-period prices are a continuous function of current prices, i.e. $p_h^2(p)$ and $p_f^2(p)$ are continuous functions.

We now interpret those assumptions which referred to anticipated second-period prices, namely, II, IV, and V, to hold for all values of p_h^2 and p_f^2 in the ranges of the anticipation functions, $p_h^2(p)$ and $p_f^2(p)$. The various functions and correspondences now depend explicitly on p, through p_f^2 and p_h^2; all the relevant continuity properties are easily seen to hold, and the existence of a compensated temporary equilibrium remains valid for elastic expectations.

Finally, to show that the compensated equilibrium is a competitive equilibrium, we need to redefine the concepts of resource-relatedness. We will say that household h' is <u>resource-related</u> to household h" for given $\bar{x}_b$ <u>and</u> p if the definition given in section 2 holds when Y_f is computed as of a fixed p_f^2 determined by p, U_h^* as of a fixed p_h^2 determined by p, and $\bar{x}_b$ is taken as given. Then household h' is said to be <u>resource-related</u> to household h" without qualification if it is so resource-related for any given $\bar{x}_b$ and p. As before, household h' is <u>indirectly resource-related</u> to household h" if there exists some chain of households, beginning with h' and ending with h", such that each household in the chain is resource-related to its successor.

VII. Every household is indirectly resource-related to every other.

With VII and the earlier assumptions, a compensated temporary equilibrium is necessarily a competitive temporary equilibrium, so the existence of competitive temporary equilibrium is established.

<u>Remark 1</u>. The theory of the firm used here is somewhere between two currently popular views. It is "managerial" in that only the expectations of managers enter into the firm' decisions; stockholders appear only as passive investors. However, in contradistinction to theories such as those of Marris [1964] and Williamson [1964], we do not ascribe to managers any motives other than profit-maximization according to their expectations.

A more general model would introduce a utility function for managers which depends in some more complicated way on the firm's production vector and current and anticipated profits; we have not investigated such a model here.

An alternative theory has the firm maximizing the current market value of its stock. That is, it chooses y_f^{12} to maximize K_f. This could be included in the present model by identifying

p_f^2 with p_h^2 for that household for which K_{hf} is a maximum, where, for each h, K_{hf} has itself been defined by maximizing over Y_f^{12} at given p and p_h^2. The only difficulty with this theory in the present framework is that as current prices change, different households value the firm most highly, and so p_f^2 might change discontinuously as p changed. This would be avoided if we assumed there is in fact a continuum of households, filling up a whole area in p_h^2-space for any given p; then p_f^2 as defined would vary continuously with p. But such a theory requires advanced methods for analysis.

Remark 2. The model here has assumed that there are no debts in the initial period, though there will, in general be debts at the beginning of the next period. If expectations are falsified, then it can happen that no equilibrium in the next period will exist without bankruptcy, because the distribution of debt which is the result of the present period's choices and therefore the initial distribution for the next period is inappropriate.

Remark 3. Of course, we are here neglecting uncertainty. This is a more serious problem than one might think; for in the presence of uncertainty it is unreasonable to assume that bonds of different firms and households are perfect substitutes; if we are not willing to assume that all individuals have the same probability distributions of prices, then it is reasonable to suppose that any firm or household has more information about matters that concern it most and therefore a household will have different subjective probability distributions for the bonds of different firms. If a given firm is then the only supplier of a commodity (its bonds) for which there are no perfect substitutes, then the capital markets cannot be assumed perfect.

Remark 4. The restriction to two periods prevents us from examining speculation in the market for shares based on other households' expectations, a matter to which Keynes [1936, pp. 154-159] has called attention in a dramatic passage. In a three-or-more-period model, a household may buy shares in a firm because he has expectations that in the second period others will have expectations which will make it profitable to sell the shares then.

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13. ABSTRACT

Existence of general equilibrium is studied for models in which the firm has a distinctive economic role. In one model, the temporary equilibrium concept of Hicks, the distinctive role of the firm is that of a forecaster; its behavior is based on expectations of future prices. In the second the firm is a price maker; a formal treatment is given of the Chamberlain model of monopolistic competition.

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